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完全正常化缔合勒让德函数及其导数与积分的递推关系

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摘　要:在地球重力场问题中,常用到完全正常化缔合勒让德函数及其导数、积分的递推关系。当前流行的地球扰动位模型均采用完全正常化的缔合勒让德函数,用此类模型可以高效方便计算各种扰动重力场元。随着本世纪多个新一代卫星重力探测计划成功实施,高阶或超高阶地球重力场模型的研究备受学界的关注。有关完全正常化缔合勒让德函数的递推关系对于高阶重力场模型具有特别意义。本文在前人研究的基础上,用初等微积分导出了若干新的递推关系式。同时还推导了正常化缔合勒让德函数及其导数、积分的检核式,这些检核式涉及地球位的球谐级数的数学性质。

关键词:完全正常化缔合勒让德函数; 导数; 积分; 递推关系; 检核式

中图法分类号:P207　　**文献标志码**:A

在地球重力场问题中,常用地球扰动位(地球实际引力位减去正常引力位)模型计算重力场元(大地水准面起伏、垂线偏差、重力异常等)。地球扰动位用球谐级数表示。缔合勒让德函数是球谐级数的组成部分。为数学表示和数值计算的方便,在实践中常规缔合勒让德函数通过乘上一个随阶 n 和次 m 而变的因子加以正常化。常见的因子有 $n!\ 2^m/(n+m)!$[1-2]、$(-1)^m n!\ /(n+m)!$[2]、$((2n+1)(n-m)!\ /4\pi(n+m)!)^{1/2}$[3]、$((2n+1)(n-m)!\ \delta_m/(n+m)!)^{1/2}$[1,4-5](若 $m=0$,$\delta_m=1$;若 $m\neq0$,$\delta_m=2$)。不同因子导致不同的正常化缔合勒让德函数。上述最后一个因子形成的缔合勒让德函数,通常被特别叫做完全正常化的缔合勒让德函数,也是当今常用的地球引力位模型所用的正常化缔合勒让德函数。

在用地球扰动位模型计算重力场元时,采用完全正常化缔合勒让德函数的递推计算显然更方便。对于重力梯度问题,还需要计算它的导数,对于格网面积平均问题,还涉及到计算它的积分,导数和积分也存在递推问题。考虑到当前流行的地球引力位模型均用完全正常化的缔合勒让德函数,研究完全正常化缔合勒让德函数及其导数和积分的递推关系就具有特别重要的意义。这方面的研究一向为学者们所关注[1-2,5]。文献[1-2]认为完全正常化缔合勒让德函数的递推计算存在数值稳定性问题,它们更倾向于上述第一个正常化因子形成的缔合勒让德函数,因而重点研究了这种所谓新的正常化勒让德函数及其导数和积分的递推问题。文献[5]是研究缔合勒让德函数的积分递推关系的代表作之一,它推导了常规的以及完全正常化的缔合勒让德函数积分的递推公式,不过仅限于单个函数的积分,而未涉及两个函数乘积的积分。本文推导了完全正常化缔合勒让德函数及其导数、积分的递推关系,不只包括单个函数的积分,还包括函数乘积的积分的递推关系,此外还推导了这些函数及其导数、积分的检核式。

1　完全正常化缔合勒让德函数及其导数的递推关系

1.1　完全正常化缔合勒让德函数的递推关系

勒让德函数定义为[4]:

$$P_n(t)=\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}(t^2-1)^n \tag{1}$$

式中,

$$t=\cos\theta \tag{2}$$

θ 代表极距。缔合勒让德函数定义为[4]:

$$P_{mn}(t)=(1-t^2)^{m/2}\frac{\mathrm{d}^m P_n(t)}{\mathrm{d}t^m} \tag{3}$$

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完全正常化缔合勒让德函数与常规缔合勒让德函数的关系被规定为[1,4-7]：

$$\bar{P}_{nm}(t)=\sqrt{\frac{(n-m)!(2n+1)\delta_m}{(n+m)!}}P_{nm}(t),$$

$$\delta_m=\begin{cases}1,m=0\\2,m>0\end{cases}\qquad(4)$$

按照式(4)，可得到完全正常化缔合勒让德函数的递推关系[1]：

$$\bar{P}_{nm}(t)=a(n,m)t\bar{P}_{n-1,m}(t)-\frac{a(n,m)}{a(n-1,m)}\bar{P}_{n-2,m}(t),0\leqslant m<n\qquad(5)$$

$$\bar{P}_{mm}(t)=b(n)(1-t^2)^{1/2}\bar{P}_{n-1,n-1}(t),n\geqslant 1\qquad(6)$$

式中，

$$a(n,m)=\sqrt{\frac{(2n+1)(2n-1)}{(n+m)(n-m)}}\qquad(7)$$

$$b(n)=\sqrt{\frac{2n+1}{2n}},\ n>1;\ b(1)=\sqrt{3}\qquad(8)$$

$$\bar{P}_{00}(t)=1,\ \bar{P}_{10}(t)=\sqrt{3}t,\ \bar{P}_{11}(t)=\sqrt{3}\ (1-t^2)^{1/2}\qquad(9)$$

1.2 完全正常化缔合勒让德函数的导数的递推关系

已知缔合勒让德函数对 θ 的导数是：

$$\frac{\mathrm{d}P_{nm}(t)}{\mathrm{d}\theta}=n\mathrm{ctan}\theta P_{nm}(t)-\frac{n+m}{\sin\theta}P_{n-1,m}(t),\quad 0\leqslant m<n,n\geqslant 1\qquad(10)$$

根据式(4)的关系，得到完全正常化缔合勒让德函数对 θ 的导数：

$$\frac{\mathrm{d}\bar{P}_{nm}(t)}{\mathrm{d}\theta}=n\mathrm{ctan}\theta\bar{P}_{nm}(t)-\frac{1}{\sin\theta}\frac{2n+1}{a(n,m)}\cdot\bar{P}_{n-1,m}(t),\quad 0\leqslant m<n,n\geqslant 1\qquad(11)$$

$$\frac{\mathrm{d}\bar{P}_{mm}(t)}{\mathrm{d}\theta}=n\mathrm{ctan}\theta\bar{P}_{mm}(t),m=n\qquad(12)$$

完全正常化缔合勒让德函数对 θ 的导数的递推关系，可以通过式(5)和式(6)对 θ 求导得到：

递推起始值：

$$\frac{\mathrm{d}\bar{P}_{nm}(t)}{\mathrm{d}\theta}=a(n,m)\left\{-(1-t^2)^{1/2}\bar{P}_{n-1,m}(t)+t\frac{\mathrm{d}\bar{P}_{n-1,m}(t)}{\mathrm{d}\theta}-\frac{1}{a(n-1,m)}\frac{\mathrm{d}\bar{P}_{n-2,m}(t)}{\mathrm{d}\theta}\right\}\ 0\leqslant m<n\qquad(13)$$

$$\frac{\mathrm{d}\bar{P}_{mm}(t)}{\mathrm{d}\theta}=b(n)\left\{t\bar{P}_{n-1,n-1}(t)+(1-t^2)^{1/2}\frac{\mathrm{d}\bar{P}_{n-1,n-1}(t)}{\mathrm{d}\theta}\right\}\qquad(14)$$

递推起始值：

$$\frac{\mathrm{d}\bar{P}_{00}(t)}{\mathrm{d}\theta}=0,\quad\frac{\mathrm{d}\bar{P}_{10}(t)}{\mathrm{d}\theta}=-\sqrt{3}\ (1-t^2)^{1/2},\ \frac{\mathrm{d}\bar{P}_{11}(t)}{\mathrm{d}\theta}=\sqrt{3}t\qquad(15)$$

2 完全正常化缔合勒让德函数的积分的递推关系

2.1 积分 $\bar{I}^s_{nm}(\theta_1,\theta_2)=\int_{\theta_1}^{\theta_2}\bar{P}_{nm}(\cos\theta)\sin\theta\mathrm{d}\theta$ 的递推关系

$$\bar{I}^s_{nm}=\frac{a(n,m)}{n+1}(1-t^2)\bar{P}_{n-1,m}(t)\bigg|_{t_1}^{t_2}+\frac{n-2}{n+1}\frac{a(n,m)}{a(n-1,m)}\bar{I}^s_{n-2,m},\quad 0\leqslant m<n\qquad(16)$$

$$\bar{I}^s_{mm}=-\frac{1}{n+1}t\bar{P}_{mm}(t)\bigg|_{t_1}^{t_2}+\frac{n}{n+1}b(n)b(n-1)\bar{I}^s_{n-2,n-2},\quad m=n\qquad(17)$$

式中，$t_1=\cos\theta_1$，$t_2=\cos\theta_2$。

递推起始值：

$$\begin{cases}\bar{I}^s_{00}=-(t_2-t_1),\quad\bar{I}^s_{10}=-\frac{\sqrt{3}}{2}(t_2^2-t_1^2)\\\bar{I}^s_{11}=\frac{\sqrt{3}}{2}\left[\arccos(t)-t\ (1-t^2)^{1/2}\right]\bigg|_{t_1}^{t_2}\end{cases}\qquad(18)$$

2.1.1 式(16)的推导

式(5)两端乘 $\sin\theta$，并对 θ 积分：

$$\int\bar{P}_{nm}(t)\sin\theta\mathrm{d}\theta=a(n,m)\int\bar{P}_{n-1,m}(t)\cos\theta\sin\theta\mathrm{d}\theta-\frac{a(n,m)}{a(n-1,m)}\int\bar{P}_{n-2,m}(t)\sin\theta\mathrm{d}\theta\qquad(19)$$

求右端第一项内的不定积分：

$$\int\bar{P}_{n-1,m}(t)\cos\theta\sin\theta\mathrm{d}\theta=\frac{1}{2}\int\bar{P}_{n-1,m}(t)\mathrm{d}(\sin^2\theta)$$

$$= \frac{1}{2}\left\{\bar{P}_{n-1,m}(t)\sin^2\theta - \int \sin^2\theta \frac{\mathrm{d}\bar{P}_{n-1,m}(t)}{\mathrm{d}\theta}\mathrm{d}\theta\right\} \quad \text{（用分部积分）}$$

$$= \frac{1}{2}\left\{\bar{P}_{n-1,m}(t)\sin^2\theta - (n-1)\int \bar{P}_{n-1,m}(t)\sin\theta\cos\theta\mathrm{d}\theta + \frac{2n-1}{a(n-1,m)}\int \bar{P}_{n-2,m}(t)\sin\theta\mathrm{d}\theta\right\}$$

这里用到了式(11)。由此式得:

$$\int \bar{P}_{n-1,m}(t)\cos\theta\sin\theta\mathrm{d}\theta = \frac{1}{n+1}\bar{P}_{n-1,m}(t)\sin^2\theta + \frac{1}{n+1}\frac{2n-1}{a(n-1,m)}\int \bar{P}_{n-2,m}(t)\sin\theta\mathrm{d}\theta \tag{20}$$

将式(20)代入式(19),并在 θ_1 和 θ_2 之间取定积分,即得式(16)。该式由文献[5]给出。

2.1.2　式(17)的推导

$$\int \bar{P}_{nm}(t)\sin\theta\mathrm{d}\theta = -\int \bar{P}_{nm}(t)\mathrm{d}\cos\theta$$

$$= -\bar{P}_{nm}(t)\cos\theta + \int \cos\theta \frac{\mathrm{d}\bar{P}_{nm}(t)}{\mathrm{d}\theta}\mathrm{d}\theta \quad \text{（用分部积分）}$$

$$= -\bar{P}_{nm}(t)\cos\theta + n\int \cos\theta\,\mathrm{ctan}\theta\bar{P}_{nm}(t)\mathrm{d}\theta \quad \text{（用式(12)）}$$

$$= -\bar{P}_{nm}(t)\cos\theta + nb(n)\int \bar{P}_{n-1,n-1}(t)\cos^2\theta\mathrm{d}\theta \ \text{（用式(6)）}$$

$$= -\bar{P}_{nm}(t)\cos\theta + nb(n)\int \bar{P}_{n-1,n-1}(t)\mathrm{d}\theta - nb(t)\int \bar{P}_{n-1,n-1}(t)\sin^2\theta\mathrm{d}\theta$$

$$= -\bar{P}_{nm}(t)\cos\theta + nb(n)b(n-1)\int \bar{P}_{n-2,n-2}(t)\sin\theta\mathrm{d}\theta - n\int \bar{P}_{nm}(t)\sin\theta\mathrm{d}\theta \quad \text{（用式(6)）}$$

由此得:

$$\int \bar{P}_{nm}(t)\sin\theta\mathrm{d}\theta = -\frac{1}{n+1}\cos\theta\bar{P}_{nm}(t) + \frac{n}{n+1}b(n)b(n-1)\int \bar{P}_{n-2,n-2}(t)\sin\theta\mathrm{d}\theta \tag{21}$$

此式在 θ_1 和 θ_2 之间取定积分,即是式(17)。

2.2　积分 $\bar{I}^c_{nm}(\theta_1,\theta_2) = \int_{\theta_1}^{\theta_2}\bar{P}_{nm}(\cos\theta)\cos\theta\mathrm{d}\theta$ 的递推关系

$$\bar{I}^c_{nm} = \frac{a(n,m)}{n+1}t(1-t^2)^{1/2}\bar{P}_{n-1,m}(t)\Big|_{t_1}^{t_2} + \frac{a(n,m)}{n+1}\bar{I}_{n-1,m} + \frac{n-2}{n+1}\frac{a(n,m)}{a(n-1,m)}\bar{I}^c_{n-2,m},$$

$$0 \leqslant m < n \tag{22}$$

$$\bar{I}^c_{nm} = \frac{b(n)}{n+1}(1-t^2)\bar{P}_{n-1,n-1}(t)\Big|_{t_1}^{t_2}, \qquad m = n \tag{23}$$

递推起始值:

$$\bar{I}^c_{00} = (1-t_2^2)^{1/2} - (1-t_1^2)^{1/2}$$

$$\bar{I}^c_{10} = \frac{\sqrt{3}}{2}\left(\arccos(t) + t(1-t^2)^{1/2}\right)\Big|_{t_1}^{t_2} \tag{24}$$

2.2.1　式(22)的推导

式(5)两端乘 $\cos\theta$,并对 θ 积分:

$$\int \bar{P}_{nm}(t)\cos\theta\mathrm{d}\theta = a(n,m)\int \bar{P}_{n-1,m}(t)\cos\theta\cos\theta\mathrm{d}\theta - \frac{a(n,m)}{a(n-1,m)}\int \bar{P}_{n-2,m}(t)\cos\theta\mathrm{d}\theta \tag{25}$$

求右端第一项中的不定积分:

$$\int \bar{P}_{n-1,m}(t)\cos\theta\cos\theta\mathrm{d}\theta = \int \bar{P}_{n-1,m}(t)\cos\theta\mathrm{d}(\sin\theta)$$

$$= \bar{P}_{n-1,m}(t)\cos\theta\sin\theta - \int \sin\theta \frac{\mathrm{d}(\bar{P}_{n-1,m}(t)\cos\theta)}{\mathrm{d}\theta}\mathrm{d}\theta \quad \text{（分部积分）}$$

$$= \bar{P}_{n-1,m}(t)\cos\theta\sin\theta - \int \sin\theta\left(-\sin\theta\bar{P}_{n-1,m}(t) + \cos\theta\frac{\mathrm{d}\bar{P}_{n-1,m}(t)}{\mathrm{d}\theta}\right)\mathrm{d}\theta$$

$$= \bar{P}_{n-1,m}(t)\cos\theta\sin\theta + \int \bar{P}_{n-1,m}(t)\sin^2\theta\mathrm{d}\theta$$

$$-(n-1)\int \bar{P}_{n-1,m}(t)\cos^2\theta\mathrm{d}\theta + \frac{2n-1}{a(n-1,m)}\int \bar{P}_{n-2,m}(t)\cos\theta\mathrm{d}\theta \quad \text{（用式(11)）}$$

$$= \bar{P}_{n-1,m}(t)\cos\theta\sin\theta + \int \bar{P}_{n-1,m}(t)\mathrm{d}\theta$$

$$- n\int \bar{P}_{n-1,m}(t)\cos^2\theta\mathrm{d}\theta + \frac{2n-1}{a(n-1,m)}\int \bar{P}_{n-2,m}(t)\cos\theta\mathrm{d}\theta$$

由此式得：

$$\int \bar{P}_{n-1,m}(t)\cos^2\theta\mathrm{d}\theta = \frac{1}{n+1}\bar{P}_{n-1,m}\cos\theta\sin\theta +$$

$$\frac{1}{n+1}\bar{I}_{n-1,m} + \frac{1}{n+1}\frac{2n-1}{a(n-1,m)}\int \bar{P}_{n-2,m}(t)\cos\theta\mathrm{d}\theta \tag{26}$$

将式(26)代入式(25)，并在 θ_1 和 θ_2 之间取定积分，即得式(22)。

2.2.2 式(23)的推导

式(6)两端乘 $\cos\theta$，并对 θ 积分：

$$\int \bar{P}_{nn}(t)\cos\theta\mathrm{d}\theta = b(n)\int \bar{P}_{n-1,n-1}(t)\sin\theta\cos\theta\mathrm{d}\theta \tag{27}$$

求右端积分项：

$$\int \bar{P}_{n-1,n-1}(t)\sin\theta\cos\theta\mathrm{d}\theta = \int \bar{P}_{n-1,n-1}(t)\sin\theta d(\sin\theta) = \bar{P}_{n-1,n-1}(t)\sin^2\theta - \int \sin\theta\frac{d(\sin\theta\bar{P}_{n-1,n-1}(t))}{\mathrm{d}\theta}\mathrm{d}\theta$$

（分部积分）

$$= \bar{P}_{n-1,n-1}(t)\sin^2\theta - \int \sin\theta\left(\cos\theta\bar{P}_{n-1,n-1}(t) + \sin\theta\frac{\mathrm{d}\bar{P}_{n-1,n-1}(t)}{\mathrm{d}\theta}\right)\mathrm{d}\theta$$

$$= \bar{P}_{n-1,n-1}(t)\sin^2\theta - \int \bar{P}_{n-1,n-1}(t)\sin\theta\cos\theta\mathrm{d}\theta - (n-1)\int \bar{P}_{n-1,n-1}(t)\sin\theta\cos\theta\mathrm{d}\theta \quad \text{（用式(11)）}$$

$$= \bar{P}_{n-1,n-1}(t)\sin^2\theta - n\int \bar{P}_{n-1,n-1}(t)\sin\theta\cos\theta\mathrm{d}\theta$$

由此式得：

$$\int \bar{P}_{n-1,n-1}(t)\sin\theta\cos\theta\mathrm{d}\theta = \frac{1}{n+1}\bar{P}_{n-1,n-1}(t)\sin^2\theta \tag{28}$$

将式(28)代入式(27)，并在 θ_1 和 θ_2 之间取定积分，即是式(23)。

2.3 积分 $\bar{I}_{nm}(\theta_1,\theta_2) = \int_{\theta_1}^{\theta_2}\bar{P}_{nm}(\cos\theta)\mathrm{d}\theta$ 的递推关系

$$\bar{I}_{nm} = a(n,m)\bar{I}^c_{n-1,m} - \frac{a(n,m)}{a(n-1,m)}\bar{I}_{n-2,m}, 0 \leqslant m < n \tag{29}$$

$$\bar{I}_{nn} = b(n)\bar{I}^s_{n-1,n-1}, m = n \tag{30}$$

递推起始值：

$$\begin{cases} \bar{I}_{00} = \arccos(t_2) - \arccos(t_1), \\ \bar{I}_{10} = \sqrt{3}((1-t_2^2)^{1/2} - (1-t_1^2)^{1/2}) \\ \bar{I}_{11} = -\sqrt{3}(t_2 - t_1) \end{cases} \tag{31}$$

式(29)和式(30)可由式(5)和式(6)两端取定积分得到。

3 完全正常化缔合勒让德函数乘积的积分的递推关系

3.1 积分 $\bar{I}^s_{nmpq}(\theta_1,\theta_2) = \int_{\theta_1}^{\theta_2}\bar{P}_{nm}(\cos\theta)\bar{P}_{pq}(\cos\theta)\sin\theta\mathrm{d}\theta$ 的递推关系

$$\bar{I}^s_{nmpq} = \frac{a(n,m)}{n+p+1}\left[(1-t^2)\bar{P}_{n-1,m}(t)\bar{P}_{pq}(t)\Big|_{t_1}^{t_2} + \frac{2p+1}{a(p,q)}\bar{I}^s_{n-1,m,p-1,q} + \frac{n-p-2}{a(n-1,m)}\bar{I}^s_{n-2,mpq}\right],$$

$$0 \leqslant m < n \text{ 且 } p \neq q \tag{32}$$

$$\bar{I}^s_{nmpq} = \frac{b(n)b(n-1)}{n+p+1}\left[-t(1-t^2)\bar{P}_{n-2,n-2}(t)\bar{P}_{pq}(t)\Big|_{t_1}^{t_2} - \frac{2p+1}{a(p,q)a(p-1,q)}\bar{I}^s_{n-2,n-2,p-2,q}\right.$$

$$\left. + \left(n+p-\frac{2p+1}{a(p,q)a(p,q)}\right)\bar{I}^s_{n-2,n-2,pq}\right], \quad m = n \text{ 且 } p \neq q \tag{33}$$

或者

$$\bar{I}^s_{mpq} = \frac{a(p,q)}{n+p+1}\left[(1-t^2)\bar{P}_m(t)\bar{P}_{p-1,q}(t)\Big|_{t_1}^{t_2} + \frac{p-n-2}{a(p-1,q)}\bar{I}^s_{m,p-2,q}\right], m = n \text{ 且 } p \neq q \tag{34}$$

$$\bar{I}^s_{mpp} = \frac{b(n)b(n-1)}{n+p+1}\left[-t(1-t^2)\bar{P}_{n-2,n-2}(t)\bar{P}_{pp}(t)\Big|_{t_1}^{t_2} + (n+p)\bar{I}^s_{n-2,n-2,pp}\right],$$

$$m = n, \text{且 } p = q \tag{35}$$

递推起始值:

$$\begin{cases} \bar{I}^s_{0000} = -(t_2 - t_1), \\ \bar{I}^s_{1100} = \frac{\sqrt{3}}{2}\left(\arccos t - t\sqrt{1-t^2}\right)\Big|_{t_1}^{t_2} \\ \bar{I}^s_{1111} = -(3t - t^3)\Big|_{t_1}^{t_2} \end{cases} \tag{36}$$

3.1.1　式(32)的推导

根据式(16),可写出:

$$\begin{aligned}
&\int \bar{P}_{nm}(t)\bar{P}_{pq}(t)\sin\theta d\theta \\
&= \int \bar{P}_{pq}(t) d\left[\frac{a(n,m)}{n+1}(1-t^2)\bar{P}_{n-1,m}(t) + \frac{n-2}{n+1}\frac{a(n,m)}{a(n-1,m)}\int \bar{P}_{n-2,m}(t)\sin\theta d\theta\right] \\
&= \frac{a(n,m)}{n+1}\int \bar{P}_{pq}(t) d[(1-t^2)\bar{P}_{n-1,m}(t)] + \frac{n-2}{n+1}\frac{a(n,m)}{a(n-1,m)}\int \bar{P}_{pq}(t)\bar{P}_{n-2,m}(t)\sin\theta d\theta \\
&= \frac{a(n,m)}{n+1}\left\{\bar{P}_{pq}(t)(1-t^2)\bar{P}_{n-1,m}(t) - \int (1-t^2)\bar{P}_{n-1,m}(t)\frac{d\bar{P}_{pq}(t)}{d\theta}d\theta\right\} + \\
&\quad \frac{n-2}{n+1}\frac{a(n,m)}{a(n-1,m)}\int \bar{P}_{pq}(t)\bar{P}_{n-2,m}(t)\sin\theta d\theta
\end{aligned} \tag{37}$$

先用式(11)代替该式中 $d\bar{P}_{pq}(t)/d\theta$,然后再将

$$t\bar{P}_{n-1,m}(t) = \frac{\bar{P}_{nm}(t)}{a(n,m)} + \frac{\bar{P}_{n-2,m}(t)}{a(n-1,m)} \quad (\text{见式}(5)) \tag{38}$$

代替所得式中的 $t\bar{P}_{n-1,m}(t)$,整理后即得:

$$\begin{aligned}
&\left(1 + \frac{p}{n+1}\right)\int \bar{P}_{nm}(t)\bar{P}_{pq}(t)\sin\theta d\theta \\
&= \frac{a(n,m)}{n+1}(1-t^2)\bar{P}_{pq}(t)\bar{P}_{n-1,m}(t) + \frac{2p+1}{n+1}\frac{a(n,m)}{a(p,q)}\int \bar{P}_{n-1,m}(t)\bar{P}_{p-1,q}(t)\sin\theta d\theta \\
&+ \left(\frac{n-2}{n+1} - \frac{p}{n+1}\right)\frac{a(n,m)}{a(n-1,m)}\int \bar{P}_{n-2,m}(t)\bar{P}_{pq}(t)\sin\theta d\theta
\end{aligned}$$

由此

$$\begin{aligned}
&\int \bar{P}_{nm}(t)\bar{P}_{pq}(t)\sin\theta d\theta \\
&= \frac{a(n,m)}{n+p+1}\left[(1-t^2)\bar{P}_{n-1,m}(t)\bar{P}_{pq}(t) + \frac{2p+1}{a(p,q)}\int \bar{P}_{n-1,m}(t)\bar{P}_{p-1,q}(t)\sin\theta d\theta\right. \\
&\quad \left. + \frac{n-p-2}{a(n-1,m)}\int \bar{P}_{n-2,m}(t)\bar{P}_{pq}(t)\sin\theta d\theta\right]
\end{aligned} \tag{39}$$

此式在 θ_1 和 θ_2 之间取定积分,即是式(32)。该式参见文献[1]。

3.1.2　式(33)的推导

根据式(17),可写出:

$$\int \bar{P}_m(t)\bar{P}_{pq}(t)\sin\theta d\theta = \int \bar{P}_{pq}(t) d\left[-\frac{t}{n+1}\bar{P}_m(t)\right] + \frac{n}{n+1}b(n)b(n-1)\int \bar{P}_{n-2,n-2}(t)\bar{P}_{pq}(t)\sin\theta d\theta$$

$$= \frac{n}{n+1}b(n)b(n-1)\int \bar{P}_{n-2,n-2}(t)\bar{P}_{pq}(t)\sin\theta d\theta - \frac{t}{n+1}\bar{P}_m(t)\bar{P}_{pq}(t) + \frac{1}{n+1}\int t\bar{P}_m(t)\frac{d\bar{P}_{pq}(t)}{d\theta}d\theta$$

$$
\begin{aligned}
&= \frac{n}{n+1} b(n)b(n-1)\int \bar{P}_{n-2,n-2}(t)\bar{P}_{pq}(t)\sin\theta \mathrm{d}\theta - \frac{t}{n+1}\bar{P}_{nm}(t)\bar{P}_{pq}(t) \\
&\quad + \frac{p}{n+1}\int \bar{P}_{nm}(t)\bar{P}_{pq}(t)\frac{\cos^2\theta}{\sin\theta}\mathrm{d}\theta - \frac{2p+1}{n+1}\frac{1}{a(p,q)}\int \bar{P}_{nm}(t)\bar{P}_{p-1,q}(t)\frac{\cos\theta}{\sin\theta}\mathrm{d}\theta \quad \left(\text{其中}\frac{\mathrm{d}\bar{P}_{pq}(t)}{\mathrm{d}\theta}\text{用式(11)}\right) \\
&= \frac{n}{n+1} b(n)b(n-1)\int \bar{P}_{n-2,n-2}(t)\bar{P}_{pq}(t)\sin\theta \mathrm{d}\theta - \frac{t}{n+1}\bar{P}_{nm}(t)\bar{P}_{pq}(t) \\
&\quad + \frac{p}{n+1} b(n)\int \bar{P}_{n-1,n-1}(t)\bar{P}_{pq}(t)\cos^2\theta \mathrm{d}\theta - \frac{2p+1}{n+1}\frac{b(n)}{a(p,q)}\int \bar{P}_{n-1,n-1}(t)\bar{P}_{p-1,q}(t)\cos\theta \mathrm{d}\theta \\
&= \frac{n}{n+1} b(n)b(n-1)\int \bar{P}_{n-2,n-2}(t)\bar{P}_{pq}(t)\sin\theta \mathrm{d}\theta - \frac{t(1-t^2)b(n)b(n-1)}{n+1}\bar{P}_{n-2,n-2}(t)\bar{P}_{pq}(t) \\
&\quad + \frac{p}{n+1} b(n)b(n-1)\int \bar{P}_{n-2,n-2}(t)\bar{P}_{pq}(t)\sin\theta \mathrm{d}\theta - \frac{p}{n+1}\int \bar{P}_{nm}(t)\bar{P}_{pq}(t)\sin\theta \mathrm{d}\theta \\
&\quad - \frac{1}{n+1}\frac{2p+1}{a(p,q)} b(n)b(n-1)\int \bar{P}_{n-2,n-2}(t)\left(\frac{\bar{P}_{pq}(t)}{a(p,q)} + \frac{\bar{P}_{p-2,q}(t)}{a(p-1,q)}\right)\sin\theta \mathrm{d}\theta \quad (\text{其中 } t\bar{P}_{n,m}(t) \text{ 用式(38)})
\end{aligned}
$$

由此式得：

$$
\begin{aligned}
&\left(1+\frac{p}{n+1}\right)\int \bar{P}_{nm}(t)\bar{P}_{pq}(t)\sin\theta \mathrm{d}\theta \\
&\quad = -\frac{t(1-t^2)b(n)b(n-1)}{n+1}\bar{P}_{n-2,n-2}(t)\bar{P}_{pq}(t) + \frac{b(n)b(n-1)}{n+1}\left(n+p-\frac{2p+1}{a(p,q)a(p,q)}\right)\cdot \\
&\quad \int \bar{P}_{n-2,n-2}(t)\bar{P}_{pq}(t)\sin\theta \mathrm{d}\theta - \frac{2p+1}{n+1}\frac{b(n)b(n-1)}{a(p,q)a(p-1,q)}\int \bar{P}_{n-2,n-2}(t)\bar{P}_{p-2,q}(t)\sin\theta \mathrm{d}\theta
\end{aligned} \tag{40}
$$

于是得：

$$
\begin{aligned}
&\int \bar{P}_{nm}(t)\bar{P}_{pq}(t)\sin\theta \mathrm{d}\theta \\
&\quad = \frac{b(n)b(n-1)}{n+p+1}\Bigg[-t(1-t^2)\bar{P}_{n-2,n-2}(t)\bar{P}_{pq}(t) + \left(n+p-\frac{2p+1}{a(p,q)a(p,q)}\right)\cdot \\
&\quad \int \bar{P}_{n-2,n-2}(t)\bar{P}_{pq}(t)\sin\theta \mathrm{d}\theta - \frac{2p+1}{a(p,q)a(p-1,q)}\int \bar{P}_{n-2,n-2}(t)\bar{P}_{p-2,q}(t)\sin\theta \mathrm{d}\theta\Bigg]
\end{aligned} \tag{41}
$$

此式在 θ_1 和 θ_2 之间取定积分，即为式(33)。

特别地，当 $p=q$ 时，$a(p,q)=\infty$，式(41)变为：

$$
\begin{aligned}
&\int \bar{P}_{nm}(t)\bar{P}_{pp}(t)\sin\theta \mathrm{d}\theta \\
&= \frac{b(n)b(n-1)}{n+p+1}\left[-t(1-t^2)\bar{P}_{n-2,n-2}(t)\bar{P}_{pp}(t) + (n+p)\int \bar{P}_{n-2,n-2}(t)\bar{P}_{pp}(t)\sin\theta \mathrm{d}\theta\right]
\end{aligned} \tag{42}
$$

此式在 θ_1 和 θ_2 之间取定积分，即为式(35)。

式(34)可由式(32)用符号互换 $p\leftrightarrow n$ 和 $q\leftrightarrow m$，并令 $m=n$ 得到。

3.2 积分 $\bar{I}^c_{nmpq}(\theta_1,\theta_2)=\int_{\theta_1}^{\theta_2}\bar{P}_{nm}(\cos\theta)\bar{P}_{pq}(\cos\theta)\cos\theta \mathrm{d}\theta$ 的递推关系

$$
\begin{aligned}
\bar{I}^c_{nmpq} &= \frac{a(n,m)}{n+p+1}\left[t(1-t^2)^{1/2}\bar{P}_{n-1,m}(t)\bar{P}_{pq}(t)\Big|_{t_1}^{t_2} + \frac{2p+1}{a(p,q)}\bar{I}^c_{n-1,m,p-1,q}\right. \\
&\quad \left. + \frac{n-p-2}{a(n-1,m)}\bar{I}^c_{n-2,mpq} + \bar{I}_{n-1,mpq}\right], \quad 0\leqslant m<n \text{，且 } p\neq q
\end{aligned} \tag{43}
$$

$$
\bar{I}^c_{nmpq} = \frac{b(n)}{n+p+1}\left[(1-t^2)\bar{P}_{n-1,n-1}(t)\bar{P}_{pq}(t)\Big|_{t_1}^{t_2} + \frac{2p+1}{a(p,q)}\bar{I}^s_{n-1,n-1,p-1,q}\right], \ m=n \text{，且 } p\neq q \tag{44}
$$

或者

$$
\bar{I}^c_{nmpq} = \frac{a(p,q)}{n+p+1}\left[t(1-t^2)^{1/2}\bar{P}_{nm}(t)\bar{P}_{p-1,q}(t)\Big|_{t_1}^{t_2} + \frac{p-n-2}{a(p-1,q)}\bar{I}^c_{p-2,q,nm} + \bar{I}_{nm,p-1,q}\right], \ m=n \text{，且 } p\neq q \tag{45}
$$

$$\bar{I}^c_{mpp}=\frac{b(n)}{n+p+1}(1-t^2)\bar{P}_{n-1,n-1}(t)\bar{P}_{pp}(t)\Big|_{t_1}^{t_2}\quad,\quad m=n,\text{且 } p=q \tag{46}$$

递推起始值:

$$\begin{cases}\bar{I}^c_{0000}=\sqrt{1-t_2^2}-\sqrt{1-t_1^2}\\ \bar{I}^c_{1100}=-\frac{\sqrt{3}}{2}(t_2^2-t_1^2)\\ \bar{I}^c_{1111}=(1-t_2^2)^{3/2}-(1-t_1^2)^{3/2}\end{cases} \tag{47}$$

3.2.1　式(43)的推导

根据式(22),可写出:

$$\int\bar{P}_{nm}(t)\bar{P}_{pq}(t)\cos\theta\mathrm{d}\theta$$
$$=\int\bar{P}_{pq}(t)\mathrm{d}\left[\frac{a(n,m)}{n+1}t(1-t^2)^{1/2}\bar{P}_{n-1,m}(t)+\frac{n-2}{n+1}\frac{a(n,m)}{a(n-1,m)}\int\bar{P}_{n-2,m}(t)\cos\theta\mathrm{d}\theta+\right.$$
$$\left.\frac{a(n,m)}{n+1}\int\bar{P}_{n-1,m}(t)\mathrm{d}\theta\right]=$$
$$\frac{a(n,m)}{n+1}\int\bar{P}_{pq}(t)\mathrm{d}[t(1-t^2)^{1/2}\bar{P}_{n-1,m}(t)]+\frac{n-2}{n+1}\frac{a(n,m)}{a(n-1,m)}\cdot$$
$$\int\bar{P}_{pq}(t)\bar{P}_{n-2,m}(t)\cos\theta\mathrm{d}\theta+\frac{a(n,m)}{n+1}\int\bar{P}_{pq}(t)\bar{P}_{n-1,m}(t)\mathrm{d}\theta$$
$$=\frac{a(n,m)}{n+1}\left\{\bar{P}_{pq}(t)t(1-t^2)^{1/2}\bar{P}_{n-1,m}(t)-\int t(1-t^2)^{1/2}\bar{P}_{n-1,m}(t)\frac{\mathrm{d}\bar{P}_{pq}(t)}{\mathrm{d}\theta}\mathrm{d}\theta\right\}+$$
$$\frac{n-2}{n+1}\frac{a(n,m)}{a(n-1,m)}\int\bar{P}_{pq}(t)\bar{P}_{n-2,m}(t)\cos\theta\mathrm{d}\theta+\frac{a(n,m)}{n+1}\int\bar{P}_{pq}(t)\bar{P}_{n-1,m}(t)\mathrm{d}\theta$$

利用式(11)和式(38),得:

$$\int\bar{P}_{nm}(t)\bar{P}_{pq}(t)\cos\theta\mathrm{d}\theta$$
$$=\frac{a(n,m)}{n+p+1}\left[t(1-t^2)^{1/2}\bar{P}_{n-1,m}(t)\bar{P}_{pq}(t)+\frac{2p+1}{a(p,q)}\int\bar{P}_{n-1,m}(t)\bar{P}_{p-1,q}(t)\cos\theta\mathrm{d}\theta+\right.$$
$$\left.\frac{n-p-2}{a(n-1,m)}\int\bar{P}_{n-2,m}(t)\bar{P}_{pq}(t)\cos\theta\mathrm{d}\theta+\int\bar{P}_{n-1,m}(t)\bar{P}_{pq}(t)\mathrm{d}\theta\right] \tag{48}$$

此式在 θ_1 和 θ_2 之间取定积分,即为式(43)。

3.2.2　式(44)的推导

根据式(23),可写出:

$$\int\bar{P}_{mm}(t)\bar{P}_{pq}(t)\cos\theta\mathrm{d}\theta=\int\bar{P}_{pq}(t)\mathrm{d}\left[\frac{b(n)}{n+1}(1-t^2)\bar{P}_{n-1,n-1}(t)\right]$$
$$=\frac{b(n)}{n+1}\left[\bar{P}_{pq}(t)(1-t^2)\bar{P}_{n-1,n-1}(t)-\int(1-t^2)\bar{P}_{n-1,n-1}(t)\frac{\mathrm{d}\bar{P}_{pq}(t)}{\mathrm{d}\theta}\mathrm{d}\theta\right]$$
$$=\frac{b(n)}{n+1}\left[(1-t^2)\bar{P}_{n-1,n-1}(t)\bar{P}_{pq}(t)-p\int\bar{P}_{n-1,n-1}(t)\bar{P}_{pq}(t)\sin\theta\cos\theta\mathrm{d}\theta\right.$$
$$\left.+\frac{2p+1}{a(p,q)}\int\bar{P}_{n-1,n-1}(t)\bar{P}_{p-1,q}(t)\sin\theta\mathrm{d}\theta\right]\quad\left(\frac{\mathrm{d}\bar{P}_{pq}(t)}{\mathrm{d}\theta}\text{ 用式(11)}\right)$$

上式右端第二项利用式(6)进行变换,稍经整理,得到:

$$\left(1+\frac{p}{n+1}\right)\int\bar{P}_{mm}(t)\bar{P}_{pq}(t)\cos\theta\mathrm{d}\theta$$
$$=\frac{b(n)}{n+1}\left[(1-t^2)\bar{P}_{n-1,n-1}(t)\bar{P}_{pq}(t)+\frac{2p+1}{a(p,q)}\int\bar{P}_{n-1,n-1}(t)\bar{P}_{p-1,q}(t)\sin\theta\mathrm{d}\theta\right] \tag{49}$$

由此式得:

$$\int\bar{P}_{mm}(t)\bar{P}_{pq}(t)\cos\theta\mathrm{d}\theta$$

$$= \frac{b(n)}{n+P+1}\left[(1-t^2)\bar{P}_{n-1,n-1}(t)\bar{P}_{pq}(t)+\frac{2p+1}{a(p,q)}\int\bar{P}_{n-1,n-1}(t)\bar{P}_{p-1,q}(t)\sin\theta\mathrm{d}\theta\right] \tag{50}$$

特别地，当 $p=q$ 时，$a(p,q)=\infty$，(49)式成为：

$$\int\bar{P}_{nn}(t)\bar{P}_{pp}(t)\cos\theta\mathrm{d}\theta = \frac{b(n)}{n+p+1}(1-t^2)\bar{P}_{n-1,n-1}(t)\bar{P}_{pp}(t) \tag{51}$$

式(50)和式(51)在 θ_1 和 θ_2 之间取定积分，即为式(44)和式(46)。

式(45)可由式(43)用符号互换 $p\leftrightarrow n$ 和 $q\leftrightarrow m$，并令 $m=n$ 得到。

3.3 积分 $\bar{I}_{nmpq}(\theta_1,\theta_2)=\int_{\theta_1}^{\theta_2}\bar{P}_{nm}(\cos\theta)\bar{P}_{pq}(\cos\theta)\mathrm{d}\theta$ 的递推关系

$$\bar{I}_{nmpq} = a(n,m)\bar{I}^c_{n-1,mpq} - \frac{a(n,m)}{a(n-1,m)}\bar{I}_{n-2,mpq},\quad 0\leqslant m<n \tag{52}$$

$$\bar{I}_{nmpq} = b(n)\bar{I}^s_{n-1,n-1,pq},\quad m=n \tag{53}$$

递推起始值：

$$\begin{cases}\bar{I}_{0000} = \arccos(t_2)-\arccos(t_1)\\ \bar{I}_{1100} = -\sqrt{3}(t_2-t_1)\\ \bar{I}_{1111} = \dfrac{3}{2}\left(\arccos(t)-t\sqrt{1-t^2}\right)\Big|_{t_1}^{t_2}\end{cases} \tag{54}$$

式(52)和式(53)是式(29)和式(30)两端取定积分的结果。

4 完全正常化缔合勒让德函数及其导数和积分的检核式

下面给出若干检核式，它们对于检核计算完全正常化缔合勒让德函数及其导数和积分的精度是有用的。

$$\sum_{m=0}^{n}\bar{P}^2_{nm}(t)=2n+1\ [5] \tag{55}$$

$$\sum_{m=0}^{n}\bar{P}_{nm}(t)\frac{\mathrm{d}\bar{P}_{nm}(t)}{\mathrm{d}\theta}=0 \tag{56}$$

$$\begin{aligned}&\bar{I}_{nm}(\theta_1,\theta_c)+\bar{I}_{nm}(\theta_c,\theta_2)=\bar{I}_{nm}(\theta_1,\theta_2),\quad \theta_1<\theta_c<\theta_2,m=0,1,\cdots,n,n=0,1,\cdots,N\\ &\bar{I}^s_{nm}(\theta_1,\theta_c)+\bar{I}^s_{nm}(\theta_c,\theta_2)=\bar{I}^s_{nm}(\theta_1,\theta_2),\quad \theta_1<\theta_c<\theta_2,m=0,1,\cdots,n,n=0,1,\cdots,N\\ &\bar{I}^c_{nm}(\theta_1,\theta_c)+\bar{I}^c_{nm}(\theta_c,\theta_2)=\bar{I}^c_{nm}(\theta_1,\theta_2),\quad \theta_1<\theta_c<\theta_2,m=0,1,\cdots,n,n=0,1,\cdots,N\end{aligned} \tag{57}$$

$$\sum_{m=0}^{n}\bar{P}_{nm}(\cos\theta_c)\bar{I}^s_{nm}(\theta_1,\theta_2)=\sqrt{2n+1}[\cos\theta_c\bar{I}^s_{n0}(\theta_1-\theta_c,\theta_2-\theta_c)+\sin\theta_c\bar{I}^c_{n0}(\theta_1-\theta_c,\theta_2-\theta_c)] \tag{58}$$

$$\sum_{m=0}^{n}\bar{P}_{nm}(\cos\theta_c)\bar{I}^c_{nm}(\theta_1,\theta_2)=\sqrt{2n+1}[-\sin\theta_c\bar{I}^s_{n0}(\theta_1-\theta_c,\theta_2-\theta_c)+\cos\theta_c\bar{I}^c_{n0}(\theta_1-\theta_c,\theta_2-\theta_c)] \tag{59}$$

$$\begin{aligned}&\sum_{m=0}^{n}\sum_{q=0}^{p}\bar{P}_{nm}(\cos\theta_c)\bar{P}_{pq}(\cos\theta_c)\bar{I}^s_{nmpq}(\theta_1,\theta_2)\\ &=\sqrt{(2n+1)(2p+1)}[\cos\theta_c\bar{I}^s_{n0p0}(\theta_1-\theta_c,\theta_2-\theta_c)+\sin\theta_c\bar{I}^c_{n0p0}(\theta_1-\theta_c,\theta_2-\theta_c)],\theta_1\leqslant\theta_c\leqslant\theta_2\end{aligned} \tag{60}$$

$$\begin{aligned}&\sum_{m=0}^{n}\sum_{q=0}^{p}\bar{P}_{nm}(\cos\theta_c)\bar{P}_{pq}(\cos\theta_c)\bar{I}^c_{nmpq}(\theta_1,\theta_2)\\ &=\sqrt{(2n+1)(2p+1)}[-\sin\theta_c\bar{I}^s_{n0p0}(\theta_1-\theta_c,\theta_2-\theta_c)+\cos\theta_c\bar{I}^c_{n0p0}(\theta_1-\theta_c,\theta_2-\theta_c)],\theta_1\leqslant\theta_c\leqslant\theta_2\end{aligned} \tag{61}$$

检核式仅整体性地对属于同一 n 阶的所有 m 次的量($\bar{P}_{nm}$，$\bar{I}^s_{nm}$，$\bar{I}^c_{nm}$ 等)进行检核，并不能检查出某一个特别 m 次的量的具体行为表现。

下面对以上诸式给予简单证明。式(55)可由球函数的分解定理证明。对式(55)的微分，即得到式(56)。定积分的基本性质使式(57)成立。

4.1 式(58)的推导

根据球函数的分解定理，对于 n 阶 m 次的完全正常化缔合勒让德函数，可写出：

$$\sum_{m=0}^{n}\bar{P}_{nm}(\cos\theta_c)\bar{P}_{nm}(\cos\theta)=\sqrt{2n+1}\bar{P}_{n0}(\cos(\theta-\theta_c))\tag{62}$$

上式两端乘 $\mathrm{d}\cos\theta$,并在 θ_1 和 θ_2 之间取定积分:

$$\sum_{m=0}^{n}\bar{P}_{nm}(\cos\theta_c)\int_{\cos\theta_1}^{\cos\theta_2}\bar{P}_{nm}(\cos\theta)\mathrm{d}\cos\theta=\sqrt{2n+1}\int_{\cos\theta_1}^{\cos\theta_2}\bar{P}_{n0}(\cos(\theta-\theta_c))\mathrm{d}\cos\theta\tag{63}$$

注意到

$$\begin{aligned}\mathrm{d}\cos\theta&=\mathrm{d}\cos(\theta_c+\theta-\theta_c)=\mathrm{d}[\cos\theta_c\cos(\theta-\theta_c)-\sin\theta_c\sin(\theta-\theta_c)]\\&=-\cos\theta_c\sin(\theta-\theta_c)\mathrm{d}(\theta-\theta_c)-\sin\theta_c\cos(\theta-\theta_c)\mathrm{d}(\theta-\theta_c)\end{aligned}\tag{64}$$

则式(63)给出:

$$\begin{aligned}&\sum_{m=0}^{n}\bar{P}_{nm}(\cos\theta_c)\int_{\theta_1}^{\theta_2}\bar{P}_{nm}(\cos\theta)\sin\theta\mathrm{d}\theta\\&=\sqrt{2n+1}\Bigg[\cos\theta_c\int_{\theta_1}^{\theta_2}\bar{P}_{n0}(\cos(\theta-\theta_c))\sin(\theta-\theta_c)\mathrm{d}(\theta-\theta_c)\\&\quad+\sin\theta_c\int_{\theta_1}^{\theta_2}\bar{P}_{n0}(\cos(\theta-\theta_c))\cos(\theta-\theta_c)\mathrm{d}(\theta-\theta_c)\end{aligned}$$

此式即是式(58)。

式(62)两端乘 $\mathrm{d}\sin\theta$,经过类似推导,可得式(59)。

4.2　式(60)的推导

对于 p 阶 q 次缔合勒让德函数,仿照式(62),写出分解式:

$$\sum_{q=0}^{p}\bar{P}_{pq}(\cos\theta_c)\bar{P}_{pq}(\cos\theta)=\sqrt{2p+1}\bar{P}_{p0}(\cos(\theta-\theta_c))\tag{65}$$

式(62)与式(65)两端相乘得:

$$\begin{aligned}&\sum_{m=0}^{n}\sum_{q=0}^{p}\bar{P}_{nm}(\cos\theta_c)\bar{P}_{nm}(\cos\theta)\bar{P}_{pq}(\cos\theta_c)\bar{P}_{pq}(\cos\theta)\\&=\sqrt{(2n+1)(2p+1)}\bar{P}_{n0}(\cos(\theta-\theta_c))\bar{P}_{p0}(\cos(\theta-\theta_c))\end{aligned}\tag{66}$$

上式两端乘 $\mathrm{d}\cos\theta$,并在 θ_1 和 θ_2 之间取定积分:

$$\begin{aligned}&\sum_{m=0}^{n}\sum_{q=0}^{p}\int_{\cos\theta_1}^{\cos\theta_2}\bar{P}_{nm}(\cos\theta_c)\bar{P}_{nm}(\cos\theta)\bar{P}_{pq}(\cos\theta_c)\bar{P}_{pq}(\cos\theta)\mathrm{d}\cos\theta\\&=\sqrt{(2n+1)(2p+1)}\int_{\cos\theta_1}^{\cos\theta_2}\bar{P}_{n0}(\cos(\theta-\theta_c))\bar{P}_{p0}(\cos(\theta-\theta_c))\mathrm{d}\cos\theta\end{aligned}$$

将式(64)代替上式右端的 $\mathrm{d}\cos\theta$,得:

$$\begin{aligned}&\sum_{m=0}^{n}\sum_{q=0}^{p}\bar{P}_{nm}(\cos\theta_c)\bar{P}_{pq}(\cos\theta_c)\int_{\cos\theta_1}^{\cos\theta_2}\bar{P}_{nm}(\cos\theta)\bar{P}_{pq}(\cos\theta)\mathrm{d}\cos\theta\\&=\sqrt{(2n+1)(2p+1)}\Bigg[-\cos\theta_c\int_{\theta_1-\theta_c}^{\theta_2-\theta}\bar{P}_{n0}(\cos(\theta-\theta_c))\bar{P}_{p0}(\cos(\theta-\theta_c))\sin(\theta-\theta_c)\mathrm{d}(\theta-\theta_c)-\\&\sin\theta_c\int_{\theta_1-\theta}^{\theta_2-\theta_1}\bar{P}_{n0}(\cos(\theta-\theta_c))\bar{P}_{p0}(\cos(\theta-\theta_c))\cos(\theta-\theta_c)\mathrm{d}(\theta-\theta_c)\Bigg]\end{aligned}$$

此式即是式(60)。

式(60)两端乘 $\mathrm{d}\sin\theta$,经与上面类似的推导,即得式(61)。

5 结 语

本文研究了有关完全正常化勒让德函数的递推关系，导出了若干新的递推关系。同时给出并证明了7种检核式，可用于检核计算的完全正常化缔和勒让德函数及导数和积分的正确性和精度估计。本文推导的递推关系式的正确性已经验证，至于它们的数值稳定性，特别是当其高阶递推时的数值稳定性，尚待进一步研究。

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Recurrence Relations for Fully Normalized Associated Legendre Functions and Their Derivatives and Integrals

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Abstract: Recurrence relations for fully normalized associated Legendre functions and their derivatives and integrals are often used when studying the Earth gravity field. Fully normalized associated Legendre functions are all adopted in current popular Earth disturbance potential models make the calculations of various disturbance gravity field elements highly efficient and easy when such models and related recurrence relations are used. New generation satellite gravity exploration missions have been successfully implemented, and high or ultrahigh degree Earth gravity field models have been quickly developed. This study area has received much attention in academic community; our investigation has a special meaning as it addresses the recurrence relations of fully normalized associated Legendre functions. Building upon previous research, several new recurrence relation expressions are derived in detail based on elementary calculus as the mathematical tool for the derivation. Formulas for assessing the values of fully normalized associated Legendre functions, their derivatives, and integrals are also deduced. This study involves the mathematical properties of the spherical harmonic expansion series of the geo-potential, thus is related to fundamental theory research.

Key words: fully normalized associated Legendre functions; derivatives; integrals; recurrence relations; check formulas

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