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三维常系数线性系统的标准基解矩阵公式

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摘 要

我们知道，对n维常系数线性系统$\dot{X}=AX$，$X(0)=\eta$；其解可表达为
$\phi(t)=\sum_{j=1}^{k}e^{\lambda_j t}\left[\sum_{i=0}^{n_j-1}\frac{t^i}{i!}(A-\lambda_j I)^i\right]\nu_j$。其中$\lambda_j$是$A$的$n_j$重特征根，$\nu_j$是$(A-\lambda_j I)^{n_j}u=0$的解空间中的元素，且$\eta=\nu_1+\cdots+\nu_k$，由此可得到线性系统的基解矩阵$\exp At$。直接使用这种方式计算不太方便，因为确定$\nu_j$的过程很繁。本文给出了三维系统的基解矩阵的直接算法，$\exp At$可直接由A的元素表出，从而为三维线性系统的分析计算带来方便。

【关键词】 常微分方程；线性系统；基解矩阵

常系数线性系统的标准基解矩阵，不论是在求解微分方程组，还是在大系统的分解、非线性系统线性化等问题中，都有着广泛的应用。本文给出的三维常系数线性系统的标准基解矩阵公式，仅只依赖于系统的系数矩阵的元素，无疑地，这给计算带来了很大的方便。同时本文的公式为利用计算机准确地计算基解矩阵提供了简捷的数学方法。

1 预备定理

1.1 三阶常系数线性系统的Cauchy问题的解公式

设
$$\left.\begin{aligned}\dot{X}&=AX\\X(0)&=\eta\end{aligned}\right\},\qquad(1)$$

其中A是3×3常数矩阵，X，η是三维列向量。我们引用［1］中的结论作为如下几个引理。

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引理 1 若λ是A的三重特征根，则（1）的解为

$$X(t)=(\eta+\beta t+\nu t^2)e^{\lambda t},$$

其中

$$\beta=(A-\lambda I)\eta \quad , \quad \nu=\frac{1}{2}(A-\lambda I)^2\eta.$$

引理 2 设λ_1是A的一个单特征根，λ_2是A的一个二重特征根，则（1）的解为

$$X(t)=\alpha e^{\lambda_1 t}+[(\eta-\alpha)+\nu t]e^{\lambda_2 t} \quad ,$$

其中

$$\alpha=\frac{1}{(\lambda_1-\lambda_2)^2}(A-\lambda_2 I)^2\eta \quad , \quad \nu=\frac{1}{\lambda_2-\lambda_1}(A-\lambda_2 I)(A-\lambda_1 I)\eta \quad .$$

引理 3 设λ_1，$\lambda_2=\sigma+\delta i$，$\lambda_3=\sigma-\delta i$是$A$的三个特征根，则（1）的解为

$$X(t)=\alpha e^{\lambda_1 t}+[(\eta-\alpha)\cos\delta t+\nu\sin\delta t]e^{t\sigma},$$

其中

$$\alpha=\frac{1}{(\lambda_1-\sigma)^2+\delta^2}[(A-\sigma I)^2\eta+\delta^2\eta] \quad ,$$

$$\nu=\frac{1}{\delta}[(A-\sigma I)\eta-(\lambda_1-\sigma)\alpha] \quad .$$

引理 4 设λ_1，λ_2，λ_3是A的三个互不相同的实特征根，则（1）的解为

$$X(t)=(\eta-\beta-\nu)e^{\lambda_1 t}+\beta e^{\lambda_2 t}+\nu e^{\lambda_3 t} \quad ,$$

其中

$$\beta=\frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}(A-\lambda_1 I)(A-\lambda_3 I)\eta \quad ,$$

$$\nu=\frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}(A-\lambda_1 I)(A-\lambda_2 I)\eta \quad .$$

1.2 一元三次代数方程的解及判别法

我们知道，一元三次方程

$$x^3+ax^2+bx+c=0 \tag{2}$$

的解为

$$\left.\begin{aligned} x_1&=u+v+a/3\\ x_2&=-(1/2)(u+v)+(\sqrt{3}/2)(u-v)i+a/3\\ x_3&=-(1/2)(u+v)-(\sqrt{3}/2)(u-v)i+a/3\end{aligned}\right\} \tag{3}$$

其中

$$u=\sqrt[3]{-q/2+\sqrt{(q/2)^2+(p/3)^3}} \quad ,$$

$$v=\sqrt[3]{-q/2-\sqrt{(q/2)^2+(p/3)^3}},$$

$$p=b-a^2/3 \quad , \quad q=2a^3/27-ab/3+c \quad .$$

从而容易证明如下引理 5。

引理 5 设x_1，x_2，x_3是（2）的三个根，则

1） $x_1=x_2=x_3 \Longleftrightarrow p=q=0$ ；

2） $x_1 \neq x_2 = x_3 \Longleftrightarrow (q/2)^2 + (p/3)^3 = 0$ 且 $p \cdot q \neq 0$ ；

3） x_1是（2）的实根，x_2，x_3是（2）的共轭复根$\Longleftrightarrow (q/2)^2 + (p/3)^3 > 0$；

4） x_1，x_2，x_3是（2）的互不相同的实根$\Longleftrightarrow (q/2)^2 + (p/3)^3 < 0$。

证明从略。

我们还需要用到下面两个公式：

设 $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix},$$

于是 $$(A-\lambda I)\eta = B_1(\lambda)\eta_1 + B_2(\lambda)\eta_2 + B_3(\lambda)\eta_3,$$

其中
$$B_1(\lambda) = \begin{pmatrix} a_{11}-\lambda \\ a_{21} \\ a_{31} \end{pmatrix}, \quad B_2(\lambda) = \begin{pmatrix} a_{12} \\ a_{22}-\lambda \\ a_{32} \end{pmatrix}, \quad B_3(\lambda) = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33}-\lambda \end{pmatrix}. \tag{4}$$

$$(A-\lambda I)(A-KI)\eta = A_1(\lambda,K)\eta_1 + A_2(\lambda,K)\eta_2 + A_3(\lambda,K)\eta_3,$$

其中
$$\left.\begin{aligned} A_1(\lambda,K) &= \begin{pmatrix} (a_{11}-\lambda)(a_{11}-k) + a_{12}a_{21} + a_{13}a_{31} \\ a_{21}(a_{11}+a_{22}-\lambda-K) + a_{23}a_{31} \\ a_{31}(a_{11}+a_{33}-\lambda-K) + a_{32}a_{21} \end{pmatrix}, \\ A_2(\lambda,K) &= \begin{pmatrix} a_{12}(a_{11}+a_{22}-\lambda-K) + a_{13}a_{32} \\ (a_{22}-\lambda)(a_{22}-K) + a_{21}a_{12} + a_{23}a_{32} \\ a_{32}(a_{22}+a_{33}-\lambda-K) + a_{31}a_{12} \end{pmatrix}, \\ A_3(\lambda,K) &= \begin{pmatrix} a_{13}(a_{11}+a_{33}-\lambda-K) + a_{12}a_{23} \\ a_{23}(a_{22}+a_{33}-\lambda-K) + a_{21}a_{13} \\ (a_{33}-\lambda)(a_{33}-K) + a_{31}a_{13} + a_{32}a_{23} \end{pmatrix}. \end{aligned}\right\} \tag{5}$$

2 三阶常系数线性系统标准基解矩阵公式

设有三阶常系数线性系统

$$\dot{X} = AX, \tag{6}$$

其中
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \dot{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}.$$

于是（6）的特征方程为

$$\lambda^3 - T_r A\lambda^2 + \left(\sum_{1\le\alpha\le\beta\le 3} \begin{vmatrix} a_{\alpha\alpha} & a_{\alpha\beta} \\ a_{\beta\alpha} & a_{\beta\beta} \end{vmatrix} \right) \lambda - \mathrm{Det}A = 0 . \tag{7}$$

记 $a \stackrel{\Delta}{=} -\mathrm{TrA} = -(a_{11} + a_{22} + a_{33}),$

$$b \stackrel{\Delta}{=} \sum_{1\le\alpha\le\beta\le 3} \begin{vmatrix} a_{\alpha\alpha} & a_{\alpha\beta} \\ a_{\beta\alpha} & a_{\beta\beta} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$

$$c \stackrel{\Delta}{=} \mathrm{Det}A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$p \stackrel{\Delta}{=} b - a^2/3 \ , \qquad q \stackrel{\Delta}{=} 2a^3/27 - ab/3 + c \ , \qquad \Delta^{\Delta}\ (q/2)^2 + (p/3)^3 .$$

情形 1 当$p=q=0$时，由引理5知，特征方程（7）有三重根$\lambda=a/3$，于是我们有：

定理 1 若$p=q=0$，则（6）的标准基解矩阵为

$$\phi(t) = (Y_1(t),\ Y_2(t),\ Y_3(t))e^{\lambda t}, \tag{8}$$

其中

$$Y_i(t) = e_i + B_i(\lambda)t + A_i(\lambda,\lambda)t^2/2 \ , \quad (i=1,\ 2,\ 3) \tag{9}$$

这里$e_1=(1,\ 0,\ 0)^T$，$e_2=(0,\ 1,\ 0)^T$，$e_3=(0,\ 0,\ 1)^T$。

将（9）具体写出即是

$$Y_1(t) = \begin{pmatrix} 1 + (a_{11}-\lambda)t + [(a_{11}-\lambda)^2 + a_{12}a_{21} + a_{13}a_{31}]t^2/2 \\ a_{21}t + [a_{21}(a_{11}+a_{22}-2\lambda) + a_{23}a_{31}]t^2/2 \\ a_{31}t + [a_{31}(a_{11}+a_{22}-2\lambda) + a_{32}a_{21}]t^2/2 \end{pmatrix},$$

$$Y_2(t) = \begin{pmatrix} a_{12} + [a_{12}(a_{11}+a_{22}-2\lambda) + a_{13}a_{31}]t^2/2 \\ 1 + (a_{22}-\lambda)t + [(a_{22}-\lambda)^2 + a_{21}a_{12} + a_{23}a_{32}]t^2/2 \\ a_{32}t + [a_{32}(a_{22}+a_{33}-2\lambda) + a_{31}a_{12}]t^2/2 \end{pmatrix},$$

$$Y_3(t) = \begin{pmatrix} a_{13}t + [a_{13}(a_{11}+a_{33}-2\lambda) + a_{12}a_{23}]t^2/2 \\ a_{23}t + [a_{23}(a_{22}+a_{33}-2\lambda) + a_{21}a_{13}]t^2/2 \\ 1 + (a_{33}-\lambda)t + [(a_{33}-\lambda)^2 + a_{31}a_{13} + a_{32}a_{23}]t^2/2 \end{pmatrix}.$$

证明：我们先求Cauchy问题

$$\dot{X} = AX \ , \qquad X(0) = \eta$$

的解。由引理1，分别取

$$\eta = e_1,\ e_2,\ e_3 \ ,$$

并由公式（4），（5）便可得到满足上述初始条件的Cauchy问题的解分别为

$$Y_1(t)e^{\lambda t},\ Y_2(t)e^{\lambda t},\quad Y_3(t)e^{\lambda t},$$

从而（6）的标准基解矩阵为（8）。

情形 2 若 $\Delta=0$ 且 $p\cdot q\neq 0$，此时方程（7）的根为

$$\lambda_1=-2\sqrt[3]{q/2}+a/3\quad,$$

$$\lambda_2=\lambda_3=\sqrt[3]{q/2}+a/3\quad.$$

于是由引理 2，我们有：

定理 2 若 $\Delta=0$，则（6）的标准基解矩阵为：

$$\phi(t)=(Y_1(t),\ Y_1(t),\ Y_3(t)),$$

其中

$$Y_i(t)=\frac{e^{\lambda_1 t}}{(\lambda_2-\lambda_1)^2}A_i(\lambda_2,\lambda_2)$$

$$+\frac{e^{\lambda_2 t}}{(\lambda_2-\lambda_1)^2}\left[(\lambda_2-\lambda_1)^2e_i-A_i(\lambda_2,\lambda_2)+(\lambda_2-\lambda_1)A_i(\lambda_1,\lambda_2)t\right],$$

$$(i=1,\ 2,\ 3).$$

具体写出即是：

$$Y_1(t)=\frac{e^{\lambda_1 t}}{(\lambda_2-\lambda_1)^2}\begin{pmatrix}(a_{11}-\lambda_2)^2+a_{12}a_{21}+a_{13}a_{31}\\ a_{21}(a_{11}+a_{22}-2\lambda_2)+a_{23}a_{31}\\ a_{31}(a_{11}+a_{33}-2\lambda_2)+a_{33}a_{22}\end{pmatrix}$$

$$+\frac{e^{\lambda_2 t}}{(\lambda_2-\lambda_1)^2}\left[\begin{pmatrix}(\lambda_2-\lambda_1)^2-(a_{11}-\lambda_2)^2-a_{12}a_{21}-a_{13}a_{31}\\ -a_{21}(a_{11}+a_{22}-2\lambda_2)-a_{23}a_{31}\\ -a_{31}(a_{11}+a_{33}-2\lambda_2)-a_{33}a_{22}\end{pmatrix}\right.$$

$$\left.+(\lambda_2-\lambda_1)\begin{pmatrix}(a_{11}-\lambda_1)(a_{11}-\lambda_2)+a_{12}a_{21}+a_{13}a_{31}\\ a_{21}(a_{11}+a_{22}-\lambda_1-\lambda_2)+a_{23}a_{31}\\ a_{31}(a_{11}+a_{33}-\lambda_1-\lambda_2)+a_{32}a_{21}\end{pmatrix}t\right];$$

$$Y_2(t)=\frac{e^{\lambda_1 t}}{(\lambda_2-\lambda_1)^2}\begin{pmatrix}a_{12}(a_{11}+a_{22}-2\lambda_2)+a_{13}a_{32}\\ (a_{22}-\lambda_2)^2+a_{12}a_{21}+a_{23}a_{32}\\ a_{32}(a_{22}+a_{33}-2\lambda_2)+a_{31}a_{12}\end{pmatrix}$$

$$+\frac{e^{\lambda_2 t}}{(\lambda_2-\lambda_1)^2}\left[\begin{pmatrix}-a_{12}(a_{11}+a_{22}-2\lambda_2)-a_{13}a_{32}\\ (\lambda_2-\lambda_1)^2-(a_{22}-\lambda_2)^2-a_{12}a_{21}-a_{23}a_{32}\\ -a_{32}(a_{22}+a_{33}-2\lambda_2)-a_{31}a_{12}\end{pmatrix}\right.$$

$$\left.+(\lambda_2-\lambda_1)\begin{pmatrix}a_{12}(a_{11}+a_{22}-\lambda_1-\lambda_2)+a_{13}a_{32}\\ (a_{22}-\lambda_1)(a_{22}-\lambda_2)+a_{21}a_{12}+a_{23}a_{32}\\ a_{32}(a_{22}+a_{33}-\lambda_1-\lambda_2)+a_{31}a_{12}\end{pmatrix}t\right];$$

$$Y_3(t)=\frac{e^{\lambda_1 t}}{(\lambda_2-\lambda_1)^2}\begin{pmatrix} a_{13}(a_{11}+a_{33}-2\lambda_2)+a_{12}a_{23}\\ a_{23}(a_{22}+a_{33}-2\lambda_2)+a_{21}a_{13}\\ (a_{33}-\lambda_2)^2+a_{31}a_{13}+a_{32}a_{23}\end{pmatrix}$$

$$+\frac{e^{\lambda_2 t}}{(\lambda_2-\lambda_1)^2}\left\{\begin{pmatrix} -a_{13}(a_{11}+a_{33}-2\lambda_2)-a_{12}a_{23}\\ -a_{23}(a_{22}+a_{33}-2\lambda_2)+a_{21}a_{13}\\ (\lambda_2-\lambda_1)^2-(a_{33}-\lambda_2)^2-a_{31}a_{13}-a_{32}a_{33}\end{pmatrix}\right.$$

$$\left.+(\lambda_2-\lambda_1)\begin{pmatrix} a_{13}(a_{11}+a_{33}-\lambda_1-\lambda_2)+a_{12}a_{23}\\ a_{23}(a_{22}+a_{33}-\lambda_1-\lambda_2)+a_{21}a_{13}\\ (a_{33}-\lambda_1)(a_{33}-\lambda_2)+a_{31}a_{13}+a_{32}a_{23}\end{pmatrix}t\right\}.$$

情形 3 当$\Delta>0$时，特征方程（7）有一实根及一对共轭复根，此时

$$\lambda_1=u+v+a/3\quad,\quad\lambda_{2,3}=\sigma\pm\sigma i\quad,$$

其中 $\sigma=-(u+v)/2+a/3,\ \delta=\sqrt{3}(u-v)/2$ ，

且 $\lambda_1-\sigma=3(u+v)/2$ ，

$$(\lambda_1-\sigma)^2+\delta^2=3[3(u+v)^2+(u-v)^2]/4 .$$

于是由引理 3 我们有

定理 3 若 $\Delta>0$，则（6）的标准基解矩阵为

$$\phi(t)=(Y_1(t),\ Y_2(t),\ Y_3(t))\quad,$$

其中

$$Y_i(t)=\frac{e^{\lambda_1 t}}{(\lambda_1-\sigma)^2+\delta^2}\left(A_i(\sigma,\sigma)+\delta^2 e_i\right)$$

$$+e^{\sigma t}\left\{\frac{1}{(\lambda_1-\sigma)^2+\delta^2}\left((\lambda_1-\sigma)^2e_i-A_i(\sigma,\sigma)\right)\cos\delta t\right.$$

$$\left.+\frac{\sin\delta t}{\delta}\left(B_i(\sigma)-\frac{\lambda_1-\sigma}{(\lambda_1-\sigma)^2+\delta^2}(A_1(\sigma,\sigma)-\delta^2e_i)\right)\right\}.\quad(i=1,\ 2,\ 3)$$

具体写出即是：

$$Y_1(t)=\frac{e^{\lambda_1 t}}{(\lambda_1-\sigma)^2+\delta^2}\begin{pmatrix}(a_{11}-\sigma)^2+a_{12}a_{21}+a_{13}a_{31}+\delta^2\\ a_{21}(a_{11}+a_{22}-2\sigma)+a_{23}a_{31}\\ a_{31}(a_{11}+a_{33}-2\sigma)+a_{32}a_{21}\end{pmatrix}$$

$$+e^{\sigma t}\left(\frac{\cos\delta t}{(\lambda_1-\sigma)^2+\delta^2}\begin{pmatrix}(\lambda_1-\sigma)^2-(a_{11}-\sigma)^2-a_{12}a_{21}-a_{13}a_{31}\\ -a_{21}(a_{11}+a_{22}-2\sigma)-a_{23}a_{31}\\ -a_{31}(a_{11}+a_{33}-2\sigma)-a_{32}a_{31}\end{pmatrix}\right.$$

$$+\frac{\sin\delta t}{\delta}\left[\begin{pmatrix} a_{11}-\sigma \\ a_{21} \\ a_{31} \end{pmatrix}-\frac{\lambda_1-\sigma}{(\lambda_1-\sigma)^2+\delta^2}\begin{pmatrix} (a_{11}-\sigma)^2+a_{12}a_{21}+a_{13}a_{31}+\delta^2 \\ a_{21}(a_{11}+a_{22}-2\sigma)+a_{23}a_{31} \\ a_{31}(a_{11}+a_{33}-2\sigma)+a_{32}a_{21} \end{pmatrix}\right]\Bigg\};$$

$$Y_2(t)=\frac{e^{\lambda_1 t}}{(\lambda_1-\sigma)^2+\delta^2}\begin{pmatrix} a_{12}(a_{11}+a_{22}-2\sigma)+a_{13}a_{32} \\ (a_{22}-\sigma)^2+a_{21}a_{12}+a_{23}a_{32}+\delta^2 \\ a_{32}(a_{22}+a_{33}-2\sigma)+a_{31}a_{12} \end{pmatrix}$$

$$+e^{\sigma t}\Bigg\{\frac{\cos\delta t}{(\lambda_1-\sigma)^2+\delta^2}\begin{pmatrix} -a_{12}(a_{11}+a_{22}-2\sigma)-a_{13}a_{32} \\ (\lambda_1-\sigma)^2+(a_{22}-\sigma)^2-a_{21}a_{12}-a_{23}a_{32} \\ -a_{32}(a_{22}+a_{33}-2\sigma)-a_{31}a_{12} \end{pmatrix}$$

$$+\frac{\sin\delta t}{\delta}\left[\begin{pmatrix} a_{12} \\ a_{22}-\sigma \\ a_{32} \end{pmatrix}-\frac{\lambda_1-\sigma}{(\lambda_1-\sigma)^2+\delta^2}\begin{pmatrix} a_{12}(a_{11}+a_{22}-2\sigma)+a_{13}a_{32} \\ (a_{22}-\sigma)^2+a_{21}a_{22}+a_{23}a_{32}+\delta^2 \\ a_{32}(a_{22}+a_{33}-2\sigma)+a_{31}a_{12} \end{pmatrix}\right]\Bigg\};$$

$$Y_3(t)=\frac{e^{\lambda_1 t}}{(\lambda_1-\sigma)^2+\delta^2}\begin{pmatrix} a_{13}(a_{11}+a_{33}-2\sigma)+a_{12}a_{23} \\ a_{23}(a_{22}+a_{33}-2\sigma)+a_{21}a_{13} \\ (a_{33}-\sigma)^2+a_{31}a_{13}+a_{32}a_{23}+\delta^2 \end{pmatrix}$$

$$+e^{\sigma t}\Bigg\{\frac{\cos\delta t}{(\lambda_1-\sigma)^2+\delta^2}\begin{pmatrix} -a_{13}(a_{11}+a_{33}-2\sigma)-a_{12}a_{23} \\ -a_{23}(a_{22}+a_{33}-2\sigma)-a_{21}a_{13} \\ (\lambda_1-\sigma)^2-(a_{33}-\sigma)^2-a_{31}a_{13}-a_{32}a_{23} \end{pmatrix}$$

$$+\frac{\sin\delta t}{\delta}\left[\begin{pmatrix} a_{31} \\ a_{32} \\ a_{33}-\sigma \end{pmatrix}-\frac{\lambda_1-\sigma}{(\lambda_1-\sigma)^2+\delta^2}\begin{pmatrix} a_{13}(a_{11}+a_{33}-2\sigma)+a_{12}a_{23} \\ a_{23}(a_{22}+a_{33}-2\sigma)+a_{21}a_{13} \\ (a_{33}-\sigma)^2+a_{31}a_{13}+a_{32}a_{23}+\delta^2 \end{pmatrix}\right]\Bigg\}.$$

情形 4 当 $\Delta<0$ 时，（7）有三个不同的实根，此时

$$u=\sqrt[3]{-q/2+\sqrt{-\Delta}\,i}, \qquad v=\sqrt[3]{-q/2-\sqrt{-\Delta}\,i}.$$

于是

$$\lambda_1=2\mathrm{Re}u+a/3,$$

$$\lambda_2=-\mathrm{Re}u-\sqrt{3}\,\mathrm{Im}u+a/3;$$

$$\lambda_3=-\mathrm{Re}u+\sqrt{3}\,\mathrm{Im}u+a/3;$$

并且

$$\lambda_2-\lambda_1=-3\mathrm{Re}u-\sqrt{3}\,\mathrm{Im}u,$$

$$\lambda_2-\lambda_3=-2\sqrt{3}\,\mathrm{Im}u,$$

$$\lambda_3-\lambda_1=-3\mathrm{Re}u+\sqrt{3}\,\mathrm{Im}u.$$

这里

$$\mathrm{Re}u = -\frac{p}{3}\cos\left(\frac{1}{3}\mathrm{arctg}\frac{-2\sqrt{-\Delta}}{q}\right),$$

$$\mathrm{Im}u = -\frac{p}{3}\sin\left(\frac{1}{3}\mathrm{arctg}\frac{-2\sqrt{-\Delta}}{q}\right).$$

于是由引理 4，并经过不太复杂的计算便可得到：

定理 4 若 $\Delta<0$，则（6）的标准基解矩阵为

$$\phi(t)=(Y_1(t),\ Y_2(t),\ Y_3(t),$$

其中

$$Y_i(t)=\frac{e^{\lambda_1 t}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\left[(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)e_i-A_i(\lambda_1,\lambda_3)-(\lambda_1-\lambda_2)B_i(\lambda)\right]$$

$$+\frac{e^{\lambda_2 t}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}A_i(\lambda_1,\lambda_3)+\frac{e^{\lambda_3 t}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}A_i(\lambda_1,\lambda_2),$$

$$(i=1,\ 2,\ 3).$$

具体写出即是：

$$Y_1(t)=\frac{e^{\lambda_1 t}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\begin{pmatrix}(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)-(a_{11}-\lambda_1)(a_{11}+\lambda_1-\lambda_2-\lambda_3)\\ -a_{21}(a_{11}+a_{22}-\lambda_2-\lambda_3)-a_{23}a_{31}\\ -a_{31}(a_{11}+a_{33}-\lambda_2-\lambda_3)-a_{32}a_{21}\end{pmatrix}$$

$$-a_{12}a_{21}-a_{13}a_{31}$$

$$+\frac{e^{\lambda_2 t}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\cdot\begin{pmatrix}(a_{11}-\lambda_1)(a_{11}-\lambda_3)+a_{12}a_{21}+a_{13}a_{31}\\ a_{21}(a_{11}+a_{22}-\lambda_1-\lambda_3)+a_{23}a_{31}\\ a_{31}(a_{11}+a_{33}-\lambda_1-\lambda_3)+a_{32}a_{21}\end{pmatrix}$$

$$+\frac{e^{\lambda_3 t}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\begin{pmatrix}(a_{11}-\lambda_1)(a_{11}-\lambda_2)+a_{12}a_{21}+a_{13}a_{32}\\ a_{21}(a_{11}+a_{22}-\lambda_1-\lambda_2)+a_{23}a_{31}\\ a_{31}(a_{11}+a_{33}-\lambda_1-\lambda_2)+a_{32}a_{21}\end{pmatrix};$$

$$Y_2(t)=\frac{e^{\lambda_1 t}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\begin{pmatrix}-a_{12}(a_{11}+a_{22}-\lambda_2-\lambda_3)-a_{13}a_{32}\\ (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)-(a_{22}-\lambda_1)(a_{22}+\lambda_1-\lambda_2-\lambda_3)\\ -a_{32}(a_{22}+a_{33}-\lambda_2\lambda_3)-a_{31}a_{12}\end{pmatrix}$$

$$
-a_{21}a_{12}-a_{23}a_{32} \Bigg)
$$

$$
+\frac{e^{\lambda_2 t}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\begin{pmatrix} a_{12}(a_{11}+a_{22}-\lambda_1-\lambda_3)a_{13}a_{32} \\ (a_{22}-\lambda_1)(a_{22}-\lambda_2)+a_{21}a_{12}+a_{23}a_{32} \\ a_{32}(a_{22}+a_{33}-\lambda_1-\lambda_3)+a_{31}a_{32} \end{pmatrix}
$$

$$
+\frac{e^{\lambda_3 t}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\begin{pmatrix} a_{12}(a_{11}+a_{22}-\lambda_1-\lambda_2)+a_{13}a_{32} \\ (a_{21}-\lambda_1)(a_{22}-\lambda_2)+a_{21}a_{12}+a_{23}a_{32} \\ a_{32}(a_{22}+a_{33}-\lambda_1-\lambda_2)+a_{31}a_{32} \end{pmatrix};
$$

$$
Y_3(t)=\frac{e^{\lambda_1 t}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\Bigg(\begin{matrix} -a_{13}(a_{11}+a_{33}-\lambda_2-\lambda_3)-a_{12}a_{23} \\ -a_{23}(a_{22}+a_{33}-\lambda_2-\lambda_3)-a_{21}a_{13} \\ (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)-(a_{33}-\lambda_1)(a_{33}+\lambda_1-\lambda_2-\lambda_3) \end{matrix}
$$

$$
-a_{31}a_{13}-a_{32}a_{23}
$$

$$
+\frac{e^{\lambda_2 t}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\begin{pmatrix} a_{13}(a_{11}+a_{33}-\lambda_1-\lambda_3)+a_{12}a_{23} \\ a_{23}(a_{22}+a_{33}-\lambda_1-\lambda_3)+a_{21}a_{13} \\ (a_{33}+\lambda_1)(a_{33}-\lambda_3)+a_{31}a_{13}+a_{32}a_{23} \end{pmatrix}
$$

$$
+\frac{e^{\lambda_3 t}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\begin{pmatrix} a_{13}(a_{11}+a_{33}-\lambda_1-\lambda_2)+a_{12}a_{23} \\ a_{23}(a_{22}+a_{33}-\lambda_1-\lambda_2)+a_{21}a_{13} \\ (a_{33}-\lambda_1)(a_{33}-\lambda_2)+a_{31}a_{13}+a_{32}a_{23} \end{pmatrix}.
$$

定理 2， 3， 4 均可仿照定理 1 的证明而得到，故证明从略。

参 考 文 献

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The Formula of the Elementary Solution Matrix of the 3-Dimensional Linear System

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Abstract

It is well known that the solution of the n-dimensional constant coefficient linear system $\dot{X} = AX$, $X(0) = \eta$ can be represented as

$$\varphi(t) = \sum_{j=1}^{K} e^{\lambda_j t} \left[\sum_{i=0}^{n_j - 1} \frac{t^i}{i!} (A - \lambda_j I)^i \right] v_j \text{ .}$$

in which, for all j, λ_j is the n-th characteristic root of A, and v_j is the element of the solution space of $(A-\lambda_j I)^{n_j} u = 0$ and $\eta = v_1 + \cdots + v_k$. From this, we can get the elementary solution matrix EXPAt of the system . However, this is not a convenient method if it is used directly in computation, because to find v_j would involve a complicated process. This paper gives a direct calculation method where by to solve the problem of the 3-dimensional systems. The EXPAt can be represented by the elements of A, which will bring about convenient in calculation.

[Key words] ordinary differential equation ; linear system ; elementary solution matrix